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EE 364 - Spring 2026

Week 15

Supplemental Notes

Advice: master the iid sum

Thrm: If X and Y independent and $Z = X + Y$:

$$\boxed{f_Z = f_X * f_Y}$$

"splat" of convolution.

Prf: SIT technique (continuous case)

$$F_Z(z) \overset{\text{CDF}}{=} P[Z \le z] \overset{\text{Hyp}}{=} P[X + Y \le z]$$

$$\overset{\text{total prob}}{=} \int_{x=-\infty}^{x=\infty} P[X + Y \le z \mid X = x] \cdot f_X(x)\, dx$$

$$\overset{\text{sub}}{=} \int_{x=-\infty}^{x=\infty} P[x + Y \le z \mid X = x] \cdot f_X(x)\, dx.$$

$$\overset{\text{ind}}{=} \int_{x=-\infty}^{x=\infty} P[x + Y \le z]\, f_X(x)\, dx$$

$$= \int_{x=-\infty}^{x=\infty} P[Y \le z - x] \cdot f_X(x)\, dx.$$

$$\therefore f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{d}{dz} \int_{x=-\infty}^{x=\infty} F_Y(z-x) \cdot f_X(x)\, dx.$$

$$= \int_{x=-\infty}^{x=\infty} \frac{d}{dz} F_Y(z-x)\, f_X(x)\, dx \quad \text{(by Leibniz's Rule)}$$

$$= \int_{x=-\infty}^{x=\infty} f_Y(z-x) \cdot f_X(x)\, dx$$

$$= (\underbrace{f_Y * f_X}_{\text{convolve}})(z)$$

then plug-in z

QED

Defn: Moment Generating Function (MGF)

$$M_X(s) = E_X\left[e^{sX}\right]$$ for real s where E_X exists: $-c < s < c$

(uniform convergence in discrete case)

$s = i\omega$ $(i = \sqrt{-1})$ gives:

Defn: Characteristic function (Lévy, 1920's)

$$\phi_X(\omega) = E_X\left[e^{i\omega X}\right]$$

$|E[e^{i\omega X}]|$ $\quad E[1] \leftarrow e^{i\omega 0}$

Note $\phi_X(\omega)$ always exists: $|\phi_X(\omega)| \le E_X\left[|e^{i\omega X}|\right] = \phi_X(0) = 1$ $\quad \forall \omega \in \mathbb{R}$

Jensen Ineq. (or triangle)

Thrm:
- $E_X[X^k] = \frac{d^k}{ds^k} M_X(s)\Big|_{s=0}$ "moment generating"
- $i^k \cdot E_X[X^k] = \frac{d^k}{d\omega^k} \phi_X(\omega)\Big|_{\omega=0}$

$\therefore \mu_X = M_X'(0) \qquad \sigma_X^2 = M_X''(0) - \left(M_X'(0)\right)^2.$

Prf: $k=1$ (continuous case when M_X exists)

$$\frac{d}{ds} M_X(s)\Big|_{s=0} = \frac{d}{ds}\int_{x=-\infty}^{x=\infty} e^{sx} f_X(x)\,dx\Big|_{s=0}$$

$$= \int_{x=-\infty}^{x=\infty} \frac{d}{ds} e^{sx}\Big|_{s=0} f_X(x)\,dx$$

$$= \int_{x=-\infty}^{x=\infty} x e^{sx}\Big|_{s=0} f_X(x)\,dx = \int_{x=-\infty}^{x=\infty} x \cdot e^{0} f_X(x)\,dx$$

$$= \int_{x=-\infty}^{x=\infty} x \cdot f_X(x)\,dx = E_X[X]$$

Repeat k times for $E_X[X^k]$. Put $s = i\omega$ for ϕ_X QED

Proposition: If X and Y independent and $Z = X + Y$

$$\begin{cases} \mathcal{M}_Z = \mathcal{M}_X \mathcal{M}_Y \\ \phi_Z = \phi_X \phi_Y \end{cases}$$

n-d: $\mathcal{M}_{\sum_{k=1}^{n} X_k} = \prod_{k=1}^{n} X_k$

Prf: $\mathcal{M}_Z(s) = E_{XY}\left[e^{sX+sY}\right] = E_{XY}\left[e^{sX}e^{sY}\right]$

$$\overset{ind}{=} E_{XY}\left[e^{sX}\right] \cdot E_{XY}\left[e^{sY}\right] = E_X\left[e^{sX}\right] \cdot E_Y\left[e^{sY}\right].$$

$$= \mathcal{M}_X(s) \cdot \mathcal{M}_Y(s)$$

QED.

(first) (second)

Ex: $X \sim b(n, p)$ and $Y \sim P(\lambda)$

(1) $\therefore \mathcal{M}_X(s) = E_X\left[e^{sX}\right] \overset{b}{=} \sum_{k=0}^{n} e^{sk} \binom{n}{k} p^k (1-p)^{n-k}$

issue spot

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^s)^k (1-p)^{n-k}$$

$$\overset{BT}{=} \boxed{(1 - p + pe^s)^n}$$

(2) $\mathcal{M}_Y(s) = E_Y\left[e^{sY}\right] = \sum_{k=0}^{\infty} e^{sk} \frac{e^{-\lambda}\lambda^k}{k!}$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!}$$

$$= \boxed{e^{\lambda(e^s - 1)}}$$

Thrm: (Levy's Continuity Theorem) (use to prove CLT)

strategy: $M_{Z_n} \xrightarrow{e} M_Z \implies Z_n \xrightarrow{d} Z$

$$X_n \xrightarrow{d} X \quad \text{at points of continuity}$$

$$\text{iff} \quad \phi_{X_n} \xrightarrow{e} \phi_X \quad (\text{and } \phi_X \text{ is continuous})$$

Ex: Poisson Law $b \xrightarrow{d} p$ if $\lambda = n \cdot p$

$$\lim_{n\to\infty} M_X(s) = \lim_{n\to\infty} (1 - p + pe^s)^n \quad \text{if } X \sim b(n,p)$$

$$\overset{\lambda = np}{=} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n} + \frac{\lambda e^s}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\lambda(e^s - 1)}{n}\right)^n$$

$$= e^{\lambda(e^s - 1)} = M_Y(s) \quad \text{if } Y \sim P(\lambda)$$

$\therefore b \xrightarrow{d} p$ by Levy's Theorem

Always remember

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Thrm: If $X \sim N(\mu_x, \sigma_x^2)$: $N(0,1) \longrightarrow \begin{cases} e^{s^2/2} \\ e^{-w^2/2} \end{cases}$

memorize

$$\text{1-D} \begin{cases} M_X(s) = e^{s\mu + s^2\sigma^2/2} \\ \phi_X(w) = e^{iw\mu - w^2\sigma^2/2} \end{cases}$$

If $\underset{n\times 1}{X} \sim N(\underset{n\times 1}{\mu_x}, \underset{n\times n}{K_{xx}})$ positive semi-definite

$$\text{n-D} \begin{cases} M_X(s) = \exp\left(s^T \mu_x + \frac{s^T K_{xx} s}{2}\right) \\ \phi_X(w) = \exp\left(iw\mu_x - \frac{w^T K_{xx} w}{2}\right) \end{cases}$$

Prf: (1-D case)

$$M_X(s) = E_X\left[e^{sX}\right] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{x=-\infty}^{x=\infty} e^{sx}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad X \sim N(\mu, \sigma^2)$$

put $z = \frac{x-\mu}{\sigma}$ $\quad \therefore x = \sigma z + \mu \quad dx = \sigma\, dz$

$(x = \pm\infty \longrightarrow z = \pm\infty)$

$$\overset{z=\frac{x-\mu}{\sigma}}{=} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{z=-\infty}^{z=\infty} e^{s\sigma z + s\mu}\, e^{-z^2/2}\, \cancel{\sigma} \cdot dz$$

$$= \frac{e^{s\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2 - 2zs\sigma\right)} dz$$

$$\overset{\text{complete the square}}{=} \frac{e^{s\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(z-s\sigma)^2 - s^2\sigma^2\right]} dz.$$

since $(z-s\sigma)^2 - s^2\sigma^2 = z^2 + s^2\sigma^2 - 2zs\sigma - s^2\sigma^2 = z^2 - 2zs\sigma$

$$= e^{s\mu + \frac{s^2\sigma^2}{2}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-s\sigma)^2} dz \right)$$

put $u = z - s\sigma$ $\quad \therefore du = dz, \quad z = \pm\infty \longrightarrow u = \pm\infty$

$$\overset{u=z-s\sigma}{=} e^{s\mu + s^2\sigma^2/2} \left(\underbrace{\frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{u=\infty} e^{-u^2/2}\, du}_{=1 \text{ since pdf}} \right)$$

$$= e^{s\mu + s^2\sigma^2/2}$$

QED.

Ex: Cauchy $X \sim C(m, d)$: $\quad f_X(x) = \dfrac{1}{\pi d\left(1 + \left(\frac{x-m}{d}\right)^2\right)}$

M_X does not exist, But ϕ_X does

$$\boxed{\phi_X(\omega) = e^{i\omega m - d|\omega|}}$$

from Cauchy's Residue Theorem:
$Z \sim C(0,1)$: $\phi_Z(\omega) = e^{-|\omega|}$

Thrm: If independent $X_1, \dots, X_n$ and $X_k \sim N(\mu_k, \sigma_k^2)$ then

note: exact

$$\sum_{k=1}^{n} X_k \sim N\left(\sum_{k=1}^{n} \mu_k, \sum_{k=1}^{n} \sigma_k^2\right)$$

similarly distributed

Prf:
$$\phi_{\sum_{k=1}^{n} a_k X_k}(w) = E\left[e^{iw \sum_{k=1}^{n} X_k}\right]$$
$$= E\left[\prod_{k=1}^{n} e^{iw X_k}\right]$$
$$\overset{\text{ind}}{=} \prod_{k=1}^{n} E\left[e^{i(w)X_k}\right]$$
$$= \prod_{k=1}^{n} \phi_{X_k}(w)$$
$$= \prod_{k=1}^{n} e^{iw\mu_k - \frac{w^2}{2}\cdot\sigma_k^2}$$
$$= e^{iw\left(\sum_{k=1}^{n} \mu_k\right) - \frac{w^2}{2}\left(\sum_{k=1}^{n} \sigma_k^2\right)}$$

$$\longleftrightarrow \sum_{k=1}^{n} X_k \sim N\left(\sum_{k=1}^{n} \mu_k, \sum_{k=1}^{n} \sigma_k^2\right)$$

QED.

Q: How does $\bar{X}_n$ differ if (iid) $X_k \sim N(\mu, \sigma^2)$ vs. $X_k \sim C(m, d)$

sample mean

Ex: If iid $X_1, \dots, X_n \sim N(\mu, \sigma^2)$: $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

(exact, not CLT approximation)

Prf:
$$\phi_{\bar{X}_n}(w) = E\left[e^{iw\bar{X}_n}\right] = E\left[e^{iw\frac{1}{n}\sum_{k=1}^{n} X_k}\right]$$
$$= E\left[e^{\sum_{k=1}^{n} i\frac{w}{n}X_k}\right] = E\left[\prod_{k=1}^{n} e^{i\frac{w}{n}X_k}\right]$$
$$\overset{\text{ind}}{=} \prod_{k=1}^{n} E\left[e^{i\left(\frac{w}{n}\right)X_k}\right] = \prod_{k=1}^{n} \phi_{X_k}\left(\frac{w}{n}\right)$$
$$\overset{X \sim N(\mu, \sigma^2)}{=} \prod_{k=1}^{n} e^{i\frac{w}{n}\mu_k - \frac{w^2}{n^2}\frac{\sigma_k^2}{2}}$$
$$\overset{\text{Id.}}{=} \left(e^{i\left(\frac{w}{n}\right)\mu - \frac{w^2}{2}\left(\sigma^2/n^2\right)}\right)^n$$
$$= e^{iw\mu - \frac{w^2}{2}\frac{\sigma^2}{n}} \longleftrightarrow \bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

QED

$\therefore \ \phi_X(w) = e^{iwm - d|w|}$

Ex: If iid $X_1, \dots, X_n \sim C(m,d)$: $\bar{X}_n \sim C(m,d)$

$\therefore$ no benefit to averaging!

<u>Prf:</u> $\phi_{\bar{X}_n}(w) = \prod_{k=1}^{n} E\left[e^{i\left(\frac{w}{n}\right)X_k}\right]$ (as in previous proof)

$$= \prod_{k=1}^{n} \phi_{X_k}\left(\frac{w}{n}\right)$$

$$\overset{X_k \sim C(m,d)}{=} \prod_{k=1}^{n} e^{i\left(\frac{w}{n}\right)m - d\left|\frac{w}{n}\right|}$$

$$= e^{i\sum_{k=1}^{n}\left(\left(\frac{w}{n}\right)m - d\left|\frac{w}{n}\right|\right)}$$

$$\overset{\text{Id.}}{=} e^{i\left(\cancel{n}\cdot\frac{w}{\cancel{n}}\cdot m - \cancel{n}\cdot d\,\frac{|w|}{\cancel{n}}\right)}$$

$$= e^{iwm - d|w|} \longleftrightarrow \bar{X}_n \sim C(m,d)$$

QED

$V[\bar{X}_n] = \frac{\sigma_X^2}{n}$ requires $\sigma_X^2 < \infty$

Note: Francis Edgeworth showed in 1883 that $\sigma_X^2 < \infty$ is <u>not</u> sufficient, even when $n=2$.

Inversion (Transform) Theorem

If CDF F_X is absolutely continuous (so pdf f_X exists):

$$\boxed{f_X(w) = \frac{1}{2\pi}\int_{w=-\infty}^{w=\infty} \phi_X(w)\cdot e^{-iwx}\, dw}$$

Central Limit Theorem ★★★

Facts: $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

Thrm: If iid $X_1, X_2, \dots$ and $\sigma_x^2 < \infty$ $(\therefore |\mu_x| < \infty)$ then

$$Z_n \xrightarrow{d} Z \sim N(0,1)$$

where $Z_n = \mathrm{STD}(\overline{X}_n) = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$

$$= \mathrm{STD}\left(\sum_{k=1}^{n} X_k\right) = \frac{\sum_{k=1}^{n} X_k - n\cdot\mu}{\sqrt{n}\cdot\sigma}$$

Prf: Strategy: show $M_{Z_n} \xrightarrow{e} M_Z(s) = e^{s^2/2}$ and then Levy's theorem.

$$M_{Z_n}(s) = E\left[e^{sZ_n}\right] = E\left[e^{s\left(\frac{\sum_{k=1}^{n} X_k - n\cdot\mu}{\sqrt{n}\sigma}\right)}\right]$$

$$= E\left[e^{\sum_{k=1}^{n} \frac{s}{\sqrt{n}\sigma}(X_k-\mu)}\right]$$

$$= E\left[\prod_{k=1}^{n} e^{\frac{s}{\sqrt{n}\sigma}(X_k-\mu)}\right] \overset{\text{ind}}{=} \prod_{k=1}^{n} E\left[e^{\frac{s}{\sqrt{n}\sigma}(X_k-\mu)}\right]$$

$$= E^n\left[e^{\frac{s(X-\mu)}{\sqrt{n}\sigma}}\right]$$

$$= \left(E\left[1 + \frac{s(X-\mu)}{\sqrt{n}\sigma} + \frac{s^2(X-\mu)^2}{2n\sigma^2} + \text{H.O.T.}\right]\right)^n$$

$$= \left(1 + 0 + \frac{s^2\sigma^2}{2n\sigma^2} + \text{H.O.T.}'\right)^n$$

$$= \left(1 + \frac{s^2/2}{n} + \underline{\text{H.O.T.}'}\right)^n \xrightarrow{e} e^{s^2/2}$$

note: need to show $\text{H.O.T.}' \to 0$

since $\lim_{n\to\infty}\left(1+\frac{c_n}{n}\right)^n = e^c$ if $\lim_{n\to\infty} c_n = c$

$$= M_Z(s) \longleftrightarrow Z \sim N(0,1)$$

$\therefore Z_n \xrightarrow{d} Z$ by Levy's continuity theorem, since $e^{s^2/2}$ is continuous

QED

Gamma pdf

$X \sim \gamma(\alpha, \theta)$, $\therefore f_X(x) = \dfrac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\,\theta^{\alpha}}$ if $x > 0$ ($\alpha > 0$ and $\theta > 0$)

for $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$

$\therefore \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$ from integration by parts

3-main properties of γ r.vars

Thrm: Say $X \sim \gamma(\alpha, \theta)$

① $E[X^k] = \theta^k \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ if $k > 0$ (proved week 6)

moments

$\therefore E[X] = \alpha\theta$ $V[X] = \alpha\theta^2$

② MGF : $\mathcal{M}_X(s) = \dfrac{1}{(1-s\theta)^{\alpha}}$ if $s < \dfrac{1}{\theta}$

③ Additivity. Say $X_1, \ldots, X_n$ are independent and

same θ

$X_k \sim \gamma(\alpha_k, \theta)$. Then $X = \sum_{k=1}^{n} X_k \sim \gamma\left(\sum_{k=1}^{n} \alpha_k, \theta\right)$.

$\therefore \sum_{k=1}^{r} X_k \sim \chi^2(r)$ if iid $X_k \sim \chi^2(1)$

since $X \sim \chi^2(r)$ if $X \sim \gamma\left(\frac{r}{2}, 2\right)$

② $\mathcal{M}_X(s) = E[e^{sX}] = \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty e^{sx} x^{\alpha-1} e^{-x/\theta}\, d\theta$

$= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\theta}-s\right)}\, dx$ if $s < \frac{1}{\theta}$

$= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^{\alpha-1} e^{-x\frac{(1-s\theta)}{\theta}}\, dx$

put $u = \frac{x(1-s\theta)}{\theta} \longleftrightarrow x = \frac{\theta u}{1-s\theta}$ and $dx = \frac{\theta\, du}{1-s\theta}$

$= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_{u=0}^{u=\infty} \frac{\theta^{\alpha-1} u^{\alpha-1}}{(1-s\theta)^{\alpha-1}} \frac{e^{-u}\theta\, du}{1-s\theta}$

$= \frac{\cancel{\theta^\alpha}}{\cancel{\Gamma(\alpha)}\cancel{\theta^\alpha}} \cdot \frac{\cancel{\Gamma(\alpha)}}{(1-s\theta)^\alpha} = \frac{1}{(1-s\theta)^\alpha}$ if $s < \frac{1}{\theta}$

QED (2)

③ Say $X_k \sim \gamma(\alpha_k, \theta)$ $\therefore \mathcal{M}_{X_k}(s) \overset{(2)}{=} \frac{1}{(1-s\theta)^{\alpha_k}}$ if $s < \frac{1}{\theta}$

$\therefore \mathcal{M}_X(s) = E[e^{sX}] = E\left[e^{s\sum_{k=1}^n X_k}\right] = E\left[e^{\sum_{k=1}^n sX_k}\right]$

$= E\left[\prod_{k=1}^n e^{sX_k}\right] \overset{\text{ind}}{=} \prod_{k=1}^n E[e^{sX_k}] = \prod_{k=1}^n \mathcal{M}_{X_k}(s)$

$\overset{\gamma}{=} \prod_{k=1}^n (1-s\theta)^{-\alpha_k} = (1-s\theta)^{-\sum_{k=1}^n \alpha_k}$

$\longleftrightarrow \quad X \sim \gamma\left(\sum_{k=1}^n \alpha_k, \theta\right)$

QED (3)

CLT and χ^2

$\Gamma\left(\frac{r}{2}, 2\right)$ $\quad \mu_\gamma = \alpha\theta$ $\quad \sigma_\gamma^2 = \alpha\theta^2$

Say $X \sim \chi^2(r)$ $\quad \therefore X = \sum_{k=1}^{r} X_k$ with $\begin{cases} ① \text{ independent } X_1, \ldots, X_n \\ ② \ X_k \sim \chi^2(1) \text{ all } k \end{cases}$

$\therefore \mu_{X_k} = 1$ and $\sigma_{X_k}^2 = 2$ (since d.f. = 1)

CLT: iid $\sum_{k=1}^{r} X_k \overset{d}{\approx} N(r\mu, r\sigma^2)$ for "large" r

$\therefore X \overset{d}{\approx} N(r\mu, r\sigma^2)$ if $X \sim \chi^2(r)$ and large r

$= N(r \cdot 1, r \cdot 2)$

$\therefore X \overset{d}{\approx} N(r, 2r)$

Similarly: $\frac{1}{r} X = \frac{1}{r} \sum_{k=1}^{r} X_k = \overline{X}_r \overset{CLT}{\approx} N\left(1, \frac{2}{r}\right)$